MATHEMATICS

B.Sc Part-II, Paper III

Topic – convergency and divergency of the series (Real Analysis)

THEOREM Prove that the infinite series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$ to ∞ is convergent if $p>1$ and divergent if $p\leq 1$.

Proof. Let the given series be denoted by S. Since S is a series of positive terms, its convergence or divergence is not affected by grouping the terms in brackets in any way we please.

Case I. Let $p>1$.

Group the terms as follows:

$$1+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\cdots \text{ to } \infty.$$

Evidently, $1=1$

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p} \text{ i.e., } \frac{2}{2^p}.$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p} \text{ i.e.; } \frac{4}{4^p}$$

$$\cdots\cdots\cdots\cdots\cdots$$

Adding we obtain $S<1+\frac{2}{2^p}+\frac{4}{4^p}+\cdots$ to ∞, which is in G.P. whose common ratio is $\frac{2}{2^p}<1$ for $p>1$.

i.e. $$S<\frac{1}{1-\frac{2}{2^p}}.$$

Hence S is convergent.

Case II. Let $p=1$.

Then the given series becomes $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots$ to ∞.

Group the terms as follows:

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \text{ to } \infty:$$

obviously $1+\frac{1}{2}=1+\frac{1}{2}$

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4} \text{ i.e. } \frac{2}{4} \text{ i.e., } \frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} \text{ i.e., } \frac{4}{8} \text{ i.e., } \frac{1}{2}$$

$$\cdots\cdots\cdots\cdots\cdots$$

Adding, we get $S>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ to ∞, which is a divergent series.

Hence S is a divergent.

Case III Let $p<1$.

Then, $\frac{1}{1^p} = 1$

$\frac{1}{2^p} > \frac{1}{2}$

$\frac{1}{3^p} > \frac{1}{3}$

$\frac{1}{4^p} > \frac{1}{4}$

$\cdots \quad \cdots$

Adding, $S > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ to ∞,
which is divergent by case II.

Hence S is divergent.

EXAMPLES

1) Test the convergence of the series

$$1 + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \cdots + \frac{1+n}{1+n^2} + \cdots \text{ to } \infty.$$

Solution Let the n^{th} term of the given series be denoted by u_n.

Then $$u_n = \frac{1+n}{1+n^2}.$$

Let us consider an auxillary series whose n^{th} term $v_n = \frac{1}{n}$.

Then $$\frac{u_n}{v_n} = \frac{n+n^2}{1+n^2} = \frac{\frac{1}{n}+1}{\frac{1}{n^2}+1}$$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n}+1}{\frac{1}{n^2}+1} = 1$, which is finite and non-zero.

Therefore, by comparision test, two set series $\sum u_n$ and $\sum v_n$ will converge or diverge simultaneously.

We have $v_n = \frac{1}{n}$.

$\therefore \sum v_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ to ∞

Here $p=1$, when compared with $\sum \frac{1}{n^p}$.

Therefore $\sum v_n$ is divergent.

Hence, by comparison test, the series $\sum u_n$ under consideration is also divergent.

2) Test the convergence of the series

$$\frac{4}{2} + \frac{7}{9} + \frac{10}{28} + \frac{13}{65} + \cdots + \frac{3n+1}{n^3+1} + \cdots \text{ to } \infty.$$

Solution. Let the n^{th} term of the given series be denoted by u_n.

Then $u_n = \frac{3n+1}{n^3+1} = \frac{3+\frac{1}{n}}{n^2\left(1+\frac{1}{n^3}\right)}$

Let the n^{th} term of the auxiliary series by v_n.

Let $v_n = \frac{1}{n^2}$. Then $\frac{u_n}{v_n} = \frac{3+\frac{1}{n}}{1+\frac{1}{n^3}}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\frac{1}{n^3}} = 3$, which is finite and non-zero.

Hence, by comparision test, both the series $\sum u_n$ and $\sum v_n$ converge or diverge simultaneously.

Here $v_n = \frac{1}{n^2}$

$\therefore \sum v_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots\cdots + \frac{1}{n^2} + \cdots$ to ∞.

Here $p = 2$, which is greater than one.

Therefore $\sum v_n$ is convergent.

Hence, by comparision test, the given series $\sum u_n$ is also convergent.

3. Test the convergence of $\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^3+5n^2+6}$

Solution Let the nth term of the given series by denoted by u_n.

Then $u_n = \frac{2n^2+1}{3n^3+5n^2+6} = \frac{2+\frac{1}{n^2}}{n\left(3+\frac{5}{n}+\frac{6}{n^3}\right)}$

Let the nth term of the auxiliary series be v_n.

Let $v_n = \frac{1}{n}$.

Then $\frac{u_n}{v_n} = \frac{2+\frac{1}{n^2}}{3+\frac{5}{n}+\frac{6}{n^3}}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{3+\frac{5}{n}+\frac{6}{n^3}} = \frac{2}{3}$,

which is finite and non-zero.

Hence by comparison test, the two series $\sum u_n$ and $\sum v_n$ converge or diverge simultaneously.

Now $v_n = \frac{1}{n}$.

$\therefore \sum v_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots\cdots$ to ∞.

Here $p = 1$.

$\therefore \sum v_n$ is divergent

Hence, by comparison test the given series $\sum u_n$ is divergent.

4. Test the convergency of the series

$$\frac{2}{1} + \frac{7}{15} + \frac{12}{53} + \frac{17}{127} + \frac{22}{249} + \cdots + \frac{5n-3}{2n^3-1}$$

Solution

Here $u_n = \frac{5n-3}{2n^3-1} = \frac{1}{n^2} \cdot \frac{5-\frac{3}{n}}{2-\frac{1}{n^3}}$

Put $v_n = \frac{1}{n^2}$; then $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{5-\frac{3}{n}}{2-\frac{1}{n^3}} = \frac{5}{2}$ which is finite.

$\therefore \sum u_n$ and $\sum v_n$ converge together.

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent. Hence $\sum u_n$ is also convergent.

5. Test the convergence of the series.

$$1 + \frac{3}{5} + \frac{5}{13} + \frac{7}{25} + \frac{9}{41} + \cdots + \frac{2n-1}{2n^2-2n+1} + \cdots \text{ to } \infty$$

Solution Here $u_n = \frac{2n-1}{2n^2-2n+1}$

Let $v_n = \frac{1}{n}$.

Then $\frac{u_n}{v_n} = \frac{2n^2-n}{2n^2-2n+1} = \frac{2-\frac{1}{n}}{2-\frac{2}{n}+\frac{1}{n^2}}$

$\therefore \underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \frac{2}{2} = 1$ which is finite and non-zero.

Therefore by comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n$ i.e., $\sum \frac{1}{n}$ diverges.

Hence the given series also diverges.

6. Test the convergency of the series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$ to ∞.

Solution Here $u_n = \frac{1}{(2n-1)^2}$

nth term of $1, 3, 5, 7, \ldots$
$= a + (n-1)d$
$= 1 + (n-1).2$
$= 2n-1$

Let $v_n = \frac{1}{n^2}$.

Then $\frac{u_n}{v_n} = \frac{n^2}{(2n-1)^2} = \frac{1}{(2-\frac{1}{n})^2}$

$\therefore \underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{1}{(2-\frac{1}{n})^2} = \frac{1}{4}$

since $\underset{n\to\infty}{Lt}$ is finite and non-zero, therefore the series $\sum u_n$ and $\sum v_n$ converge or diverge together.

But the series $\sum v_n$ i.e $\sum \frac{1}{n^2}$ is convergent, Hence the given series is also convergent.